

Critical behaviour of anisotropic magnets with quenched disorder: replica symmetry breaking studied by operator product expansion

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We study critical behaviour of disordered magnets near four dimensions. We consider the system with explicit cubic anisotropy and scalar disorder and that with random direction of anisotropy axis. The quenched disorder is taken into account by replica method. Using the method of operator product expansion, we derive in the first order to ϵ approximation the renormalization group equations taking into account possible replica symmetry breaking.

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I. INTRODUCTION

Wilson and Fisher calculation of the critical exponents for the ϕ^4 model by ϵ expansion of the renormalization group (RG) equations,¹ was a groundbreaking discovery. Soon after that the ϵ expansion was used to derive the RG equations for the ϕ^4 -model with scalar quenched disorder²⁻⁴ (see also Ref. 5). Anisotropic models were studied by Aharony: first the pure ϕ^4 model⁶ (see also Refs. 7-9), and later disordered anisotropic models¹⁰.

The quenched disorder was taken into account in the mentioned above works by the replica method¹¹. This method was advanced by Dotsenko et al.¹², who have shown that the replica symmetry, assumed in the previous application of the method to the RG theory, can be spontaneously broken. (This kind of replica symmetry breaking (RSB) was previously discovered by Parisi in the theory of spin glasses¹³.)

The two models one with cubic anisotropy and quenched scalar disorder and another with random direction of the axis of anisotropy, studied previously by Aharony, is the subject of the present contribution. The RG equations are obtained in a very simple and appealing way using the operator product expansion (OPE) method, another great discovery of Wilson¹⁴ (see also Refs. 15,16; application of this method to the theory of classical phase transitions is particularly clearly presented in the book by Cardy⁹). To present the method and notation we start by rederiving by the OPE method the replica symmetric RG equations. In the most important in the paper Section V we generalize these equations to take into account the RSB.

II. OPERATOR PRODUCT EXPANSION AND THE PERTURBATIVE RENORMALIZATION GROUP

The operator product expansion is a universal conception of quantum field theory. The essential idea is that for any two local operator quantum fields at points \mathbf{x}, \mathbf{y} (we consider Euclidean space) their product may be expressed in terms of a series of local quantum fields at any other point (which may be identified with \mathbf{x} or \mathbf{y}) times c -number coefficient functions which depend on $|\mathbf{x} - \mathbf{y}|$.

This general statement, in particular case that will be relevant for us, can be presented as follows^{9,17}. Let Φ (called scaling field) be some product of massless free fields. Then

$$\begin{aligned} & : \Phi_i(\mathbf{x}) : \times : \Phi_j(\mathbf{y}) : = : \Phi_i(\mathbf{x}) \Phi_j(\mathbf{x}) : + \\ & + \sum_{1\text{-contraction } [(\mathbf{x},i)(\mathbf{y},j)]} \Delta_{AB}(|\mathbf{x} - \mathbf{y}|) : \Phi_i(\mathbf{x}) \Phi_j(\mathbf{x}) : \\ & + \sum_{2\text{-contractions } [(\mathbf{x},i)(\mathbf{y},j)], [(\mathbf{x},i')(\mathbf{y},j')]} \Delta_{AB}(|\mathbf{x} - \mathbf{y}|) \\ & \Delta_{A'B'}(|\mathbf{x} - \mathbf{y}|) : \Phi_i(\mathbf{x}) \Phi_j(\mathbf{x}) : + \dots, \end{aligned} \quad (1)$$

where $: X :$ stands for normal ordered operator X , and

$$\Delta_{AB}(x) = \frac{\delta_{AB}}{4\pi} \frac{\Gamma(\sigma)}{\pi^\sigma} \frac{1}{x^{2\sigma}}, \quad \sigma = d/2 - 1 \quad (2)$$

is the propagator of the free fields. (Further on, not to clutter notation, we'll omit the colon signs, where it can not lead to confusion.)

Let us consider a fixed point Hamiltonian H^* which is perturbed by a number of scaling fields, so that the partition function is⁹

$$Z = \text{Tr} \exp \left\{ - \int d^d \mathbf{r} \left[H^* + \sum_i a_c^{x_i} G_i \Phi_i(\mathbf{r}) \right] \right\}, \quad (3)$$

where x_i is the appropriate natural scaling dimension, and microscopic cut-off a_c is implied in the integral. Expanding in the powers of coupling we obtain

$$Z = Z^* \left[1 - \sum_i a_c^{x_i-d} G_i \int d^d \mathbf{r} \langle \Phi_i(\mathbf{r}) \rangle + \frac{1}{2} \sum_{ij} a_c^{x_i+x_j-2d} G_i G_j \int d^d \mathbf{r}_1 d^d \mathbf{r}_2 \langle \Phi_i(\mathbf{r}_1) \Phi_j(\mathbf{r}_2) \rangle - \dots \right],$$

where all correlation functions are to be evaluated with respect to the fixed point Hamiltonian H^* .

We implement the RG by changing the microscopic cut-off from a_c to $(1 + d\ell)a_c$ and asking how the couplings G_i should be changed to preserve the partition function Z . The answer is given by the perturbative RG equations⁹

$$\frac{dG_k}{d\ell} = (d - x_k)G_k - \sum_{ij} c_{kij} G_i G_j + \dots, \quad (5)$$

where summation is with respect to all pairs i, j such that Φ_k appears in the product $\Phi_i \Phi_j$, as the result of contraction(s) (there should be at least one). The coefficients c_{kij} depend upon the specific realization of renormalization procedure and typically are presented containing multipliers expressed through the area of the hypersphere of unit radius in d dimensions, π and things like this, which appear as the result of calculation of loop integrals. However, these multipliers are the same for all coefficients with the same number of contractions (loops). Because only ratio of the coefficients is important, if all the terms in Eq. (5) contain the same number of contractions (as will be in our case), we can always make all relevant c_{kij} equal to 1 by appropriate rescaling of G_i .⁹

III. QUENCHED SCALAR DISORDER AND CUBIC ANISOTROPY

A. Replica method

Consider the d -dimensional system described by the n -component order parameter $\phi_i(\mathbf{r})$ ($i = 1, 2, \dots, n$) with the symmetry explicitly broken from $O(n)$ to cubic and quenched random scalar disorder. Combining the well known results we may describe the system in vicinity of the critical points by the following continuous Hamiltonian^{5,9,11}:

$$H[\delta\tau, \phi] = \int d^D \mathbf{r} \left\{ \frac{1}{2} \sum_{a=1}^n (\nabla \phi_a(\mathbf{r}))^2 + a_c^{-2} (\tau - \delta\tau(\mathbf{r})) \sum_{a=1}^n \phi_a(\mathbf{r})^2 + u a_c^{d-4} \sum_{a,b=1}^n \phi_a(\mathbf{r})^2 \phi_b(\mathbf{r})^2 + v a_c^{d-4} \sum_{a=1}^n \phi_a(\mathbf{r})^4 \right\}, \quad (6)$$

where a_c is a microscopic cut-off, $\tau = (T - T_c)/T_c$. (It is known that fluctuations of the other two coefficients in the Landau-Ginsburg functional do not influence critical behavior for small ϵ .⁵) According to the replica method (4) one has to calculate the following partition function (fluctuations of the effective transition temperature $\delta\tau(\mathbf{r})$ we assume to be Gaussian)

$$Z_p = \overline{\left(\int \prod_{a=1}^n D\phi_a \exp\{-H[\delta\tau, \phi]\} \right)^p} = \int D\delta\tau(\mathbf{r}) \int \prod_{a=1}^n \prod_{\alpha=1}^p D\phi_a^\alpha \exp\left\{ -\frac{1}{4\Delta} \int d^d \mathbf{r} [a_c^{-d} (\delta\tau)^2 - H[\delta\tau, \phi]] \right\}, \quad (7)$$

where the superscript α labels the replicas.

As it is well known, the scheme of the replica method can be described in the following steps¹¹. First, the measurable quantities we are interested in should be calculated for integer p . Second, the analytic continuation of the obtained functions of the parameter p should be made for an arbitrary non-integer p . Finally, the limit $p \rightarrow 0$ should be taken.

After Gaussian integration over $\delta\tau(\mathbf{r})$ one gets:

$$Z_p = \int \prod_{\mathcal{A}} D\phi^{\mathcal{A}} \exp^{-\int d^d \mathbf{r} \{H_0[\phi] + H_{int}[\phi]\}}, \quad (8)$$

where

$$H_{int}[\phi] = a_c^{-2} \tau \sum_{\mathcal{A}} \Phi^{\mathcal{A}} + a_c^{d-4} \sum_{\mathcal{AB}} g_{\mathcal{AB}} \Phi^{\mathcal{AB}}, \quad (9)$$

and

$$g_{\mathcal{AB}} = (u + v\delta_{ab})\delta_{\alpha\beta} - \Delta; \quad (10)$$

calligraphic capital letter stands for a pair of replica index and vector index.

It will be convenient for us to rewrite Hamiltonian (9) as

$$H_{int} = a_c^{-2} \tau \Phi^1 + a_c^{d-4} [u \overline{\Phi} - \Delta \Phi + v \widehat{\Phi}], \quad (11)$$

where

$$\Phi^1 = \sum_{a\alpha} (\phi_a^\alpha)^2 \quad (12)$$

$$\Phi = \sum_{ab\alpha\beta} (\phi_a^\alpha)^2 (\phi_b^\beta)^2 \quad (13)$$

$$\overline{\Phi} = \sum_{ab\alpha} (\phi_a^\alpha)^2 (\phi_b^\alpha)^2 \quad (14)$$

$$\widehat{\Phi} = \sum_{a\alpha} (\phi_a^\alpha)^4. \quad (15)$$

B. Multiplication table

To derive the RG equations we will need expansion (which in the approximation used are probably better to call merging) coefficients for two types of products. These expansions can be graphically presented as follows:

$$\begin{aligned}
 (\phi_K)^2 \times (\phi_A)^2 (\phi_B)^2 &= (\phi_K \phi_K)(\phi_A \phi_A)(\phi_B \phi_B) \\
 &+ (\phi_K \phi_K)(\phi_A \phi_B)(\phi_A \phi_B) + \text{permutations}
 \end{aligned} \quad (16)$$

and

$$\begin{aligned}
 (\phi_A)^2 (\phi_B)^2 \times (\phi_C)^2 (\phi_D)^2 & \\
 &= (\phi_A \phi_A)(\phi_B \phi_B)(\phi_C \phi_C)(\phi_D \phi_D) \\
 &+ (\phi_A \phi_A)(\phi_B \phi_B)(\phi_C \phi_D)(\phi_D \phi_C) \\
 &+ (\phi_A \phi_A)(\phi_B \phi_C)(\phi_C \phi_B)(\phi_D \phi_D) + \text{permutations},
 \end{aligned} \quad (17)$$

where we have ignored the irrelevant terms⁹ and the field independent one. Permutations means the diagrams which can be obtained from the drawn ones by interchanging field operators within the brackets and/or \mathcal{A} and \mathcal{B} and/or \mathcal{C} and \mathcal{D} and/or (in the case of in Eq. (17)) \mathcal{AB} and \mathcal{CD} .

Eqs. (16), (17) allow us to obtain the multiplication table for operators (12) - (15)

$$\Phi^1 \times \Phi = 4(2 + pn)\Phi^1 \quad (18)$$

$$\Phi^1 \times \bar{\Phi} = 4(2 + n)\Phi^1 \quad (19)$$

$$\Phi^1 \times \hat{\Phi} = 12\Phi^1 \quad (20)$$

$$\Phi \times \Phi = 8(8 + pn)\Phi \quad (21)$$

$$\Phi \times \bar{\Phi} = 8(2 + n)\Phi + 48\bar{\Phi} \quad (22)$$

$$\Phi \times \hat{\Phi} = 24\Phi + 48\hat{\Phi} \quad (23)$$

$$\bar{\Phi} \times \bar{\Phi} = 8(8 + n)\bar{\Phi} \quad (24)$$

$$\bar{\Phi} \times \hat{\Phi} = 24\bar{\Phi} + 48\hat{\Phi} \quad (25)$$

$$\hat{\Phi} \times \hat{\Phi} = 72\hat{\Phi}. \quad (26)$$

(We have omitted the propagators to draw attention just to the combinatoric multipliers.)

C. RG equations

For dimension d slightly smaller than four, we can look for fixed points of the Hamiltonian (9) in the vicinity of the Gaussian one, that is consider H_{int} as a perturbation. The perturbation theory is actually expansion with respect to parameter $\epsilon = 4 - d$. In the lowest approximation with respect to this parameter we should restrict ourselves by two first terms in the r.h.s. of Eq. (5).

Thus taking into account our multiplication table (18) - (26) and substituting the expansion coefficients into Eq.

(5) we obtain the RG equations

$$\frac{du}{d\ell} = \epsilon u - 8[(8 + n)u^2 - 12u\Delta + 6uv] \quad (27)$$

$$\frac{d\Delta}{d\ell} = \epsilon\Delta - 8[(4 + 2n)u\Delta - (8 + pn)\Delta^2 + 6\Delta v] \quad (28)$$

$$\frac{dv}{d\ell} = \epsilon v - 8[12uv - 12\Delta v + 9v^2]. \quad (29)$$

and

$$\frac{d\tau}{d\ell} = 2\tau - 8[(2 + n)u - (2 + pn)\Delta + 3v]\tau. \quad (30)$$

The RG equations coincide with those of Ref. 10 (for generalization including ϵ^2 terms see Refs. 18 and 19).

IV. RANDOM DIRECTION OF THE ANISOTROPY AXIS

The random-axis model

$$H = -J \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \vec{S}_j - D_0 \sum_i (\vec{x}_i \vec{S}_i)^2 \quad (31)$$

was introduced by Harris et al.²⁰ to describe the magnetic properties of amorphous alloys. In Eq. (31) \vec{S}_i is an n -component spin vector located at the lattice site i , $\langle ij \rangle$ denotes a pair of spin sites, J_{ij} is the exchange interaction, \hat{x}_i is a unit vector which points in the local (random) direction of the uniaxial anisotropy at the site i , and D_0 is the anisotropy constant.

The Hamiltonian of the model in the continuum approximation and after the replica trick can be presented as¹⁰

$$H_{int} = a_c^{-2} \tau \sum_{\mathcal{A}} \Phi^{\mathcal{A}} + a_c^{d-4} \left[\sum_{\mathcal{AB}} g_{\mathcal{AB}} \Phi^{\mathcal{AB}} + w\Psi \right], \quad (32)$$

where

$$g_{\mathcal{AB}} = g_{ab} = u\delta_{\alpha\beta} - \Delta, \quad (33)$$

and

$$\Psi = \sum_{ab\alpha\beta} \phi_a^\alpha \phi_b^\alpha \phi_a^\beta \phi_b^\beta. \quad (34)$$

The connection between the parameters of the Hamiltonians (31) and (32) will be of no interest to us.

It is convenient to rewrite Hamiltonian (32) as

$$H_{int} = a_c^{-2} \tau \Phi^1 + a_c^{d-4} [\bar{\Phi} - \Delta\Phi + w\Psi], \quad (35)$$

where Φ^1 , $\bar{\Phi}$, Φ and Ψ were defined above.

Using Eq. (17) we obtain additional lines of the multiplication table necessary for obtaining the RG equations in the case considered.

$$\Phi^1 \times \Psi = 4(1 + p + n)\Phi^1 \quad (36)$$

$$\Phi \times \Psi = 8(1 + p + n)\Phi + 48\Psi \quad (37)$$

$$\bar{\Phi} \times \Psi = 8\Phi + 8(5 + n)\bar{\Phi} + 16\Psi \quad (38)$$

$$\Psi \times \Psi = 8(4 + p + n)\Psi + 24\Phi. \quad (39)$$

Thus we obtain the RG equations

$$\frac{du}{d\ell} = \epsilon u - 8[(8 + n)u^2 - 12u\Delta + 2(5 + n)uw] \quad (40)$$

$$\frac{d\Delta}{d\ell} = \epsilon\Delta - 8[(4 + 2n)u\Delta - (8 + pn)\Delta^2 - 2uw + 2(1 + p + n)\Delta w - 3w^2] \quad (41)$$

$$\frac{dw}{d\ell} = \epsilon w - 8[4uw - 12\Delta w + (4 + p + n)w^2]. \quad (42)$$

The analog of Eqs. (30) is

$$\frac{d\tau}{d\ell} = 2\tau - 8[(2 + n)u - (2 + pn)\Delta + (1 + p + n)w]\tau. \quad (43)$$

Eqs. (40) - (43) exactly coincide with those from Ref.¹⁰.

Again, finally we should go in Eqs. (40)-(43) to the limit $p \rightarrow 0$.

V. REPLICAS SYMMETRY BREAKING

Consider first the case of scalar disorder from Section III. In this case there is one replica non-diagonal scaling field – Φ . Hence possible RSB will be taken into account if we generalize the Hamiltonian (11) in the following way.

$$H_{int} = a_c^{-2}\tau\Phi^1 + a_c^{d-4} \left[u\bar{\Phi} - \sum_{\alpha\beta} \Delta_{\alpha\beta}\Phi_{\alpha\beta} + v\hat{\Phi} \right], \quad (44)$$

where

$$\Phi_{\alpha\beta} = \sum_{ab} (\phi_a^\alpha)^2 (\phi_b^\beta)^2. \quad (45)$$

We have to make more specific the lines containing Φ

in our multiplication table (18) - (26):

$$\Phi^1 \times \sum_{\alpha\beta} \Delta_{\alpha\beta}\Phi_{\alpha\beta} = 4 \sum_{\alpha\alpha} \left(2\Delta_{\alpha\alpha} + n \sum_{\beta} \Delta_{\alpha\beta} \right) (\phi_a^\alpha)^2 \quad (46)$$

$$\sum_{\alpha\beta} \Delta_{\alpha\beta}\Phi_{\alpha\beta} \times \sum_{\gamma\delta} \Delta_{\gamma\delta}\Phi_{\gamma\delta} = 8 \sum_{\alpha\beta} \left[4\Delta_{\alpha\beta}^2 + 2(\Delta_{\alpha\alpha} + \Delta_{\beta\beta})\Delta_{\alpha\beta} + n \sum_{\gamma} \Delta_{\alpha\gamma}\Delta_{\gamma\beta} \right] \Phi_{\alpha\beta} \quad (47)$$

$$\sum_{\alpha\beta} \Delta_{\alpha\beta}\Phi_{\alpha\beta} \times \bar{\Phi} = 8 \sum_{\alpha\beta} (2 + n) \Delta_{\alpha\beta}\Phi_{\alpha\beta} + 48 \sum_{ab\alpha} \Delta_{\alpha\alpha} (\Phi_a^\alpha)^2 (\Phi_b^\alpha)^2 \quad (48)$$

$$\sum_{\alpha\beta} \Delta_{\alpha\beta}\Phi_{\alpha\beta} \times \hat{\Phi} = 24 \sum_{\alpha\beta} \Delta_{\alpha\beta}\Phi_{\alpha\beta} + 48 \sum_{\alpha\alpha} \Delta_{\alpha\alpha} (\Phi_a^\alpha)^4. \quad (49)$$

Substituting the results from our expanded multiplication table into Eq. (5) we obtain the RG equations. We shall study these equations assuming that the matrix $\Delta_{\alpha\beta}$ has a general Parisi RSB structure, and in the limit $p \rightarrow 0$ is parameterized in terms of its diagonal elements $\tilde{\Delta}$ and the off-diagonal function $\Delta(x)$ defined in the interval $0 < x < 1$ (which can be presented as $\Delta = (\tilde{\Delta}, \Delta(x))$).¹¹ Due to such parametrization we immediately recover Eqs. (27), (29) and (30) (the last one with $p = 0$), with Δ substituted by $\tilde{\Delta}$.

Where the standard technique of the Parisi RSB algebra is substantially different from ordinary matrix algebra is product of matrices¹³ (and we have such product in the r.h.s. of Eq. (47)). The definition of the product of Parisi matrices is as follows. Let $a = (\tilde{a}, a(x))$, $b = (\tilde{b}, b(x))$, $c = (\tilde{c}, c(x))$, and $c = ab$. Then

$$\tilde{c} = \tilde{a}\tilde{b} - \int_0^1 dx a(x)b(x) \quad (50)$$

$$c(x) = \left(\tilde{a} - \int_0^1 dy a(y) \right) b(x) + \left(\tilde{b} - \int_0^1 dy b(y) \right) a(x) - \int_0^x dy [a(y) - a(x)][b(y) - b(x)]. \quad (51)$$

Thus we recover Eq. (28) (with $p = 0$), only this time Δ is not a number but a Parisi matrix. We generalized RG equations from Ref. 12 for the case of cubic anisotropy present in the model.

For the case of random anisotropy axis the situation is very much similar. In this case there are two replica non-diagonal scaling fields Φ and Ψ . Hence the Hamiltonian which takes into account possible symmetry break-

ing should be written in the following form.

$$H_{int} = a_c^{-2} \tau \Phi^1 + a_c^{d-4} \left[u \bar{\Phi} - \sum_{\alpha\beta} \Delta_{\alpha\beta} \Phi_{\alpha\beta} + \sum_{ab\alpha\beta} w_{\alpha\beta} \phi_a^\alpha \phi_b^\alpha \phi_a^\beta \phi_b^\beta \right], \quad (52)$$

where Δ and w are both Parisi matrices. Repeating the derivation from above we again recover Eqs. (40), (43) (the last one with $p = 0$), where $\tilde{\Delta}$ substitutes for Δ , and \tilde{w} substitutes for w , and Eqs. (41), (42) (with $p = 0$), where product of the matrices Δ and w is understood according to Eqs. (50) and (51).

The analysis of the fixed points of the RG equations obtained in this Section is left for consideration in future.

VI. APPENDIX

Phase transitions are described by the stable fixed points of the RG equations. In this Appendix for pedagogical purposes we present analysis of the fixed points of the replica symmetric RG equations for the quenched scalar disorder and their stability.

Eqs. (27) - (29) have 7 fixed points¹⁰:

- (i) Gaussian fixed point: $(u^*, \Delta^*, v^*) = (0, 0, 0)$
- (ii) Pure Heisenberg fixed point: $(u^*, \Delta^*, v^*) = (\epsilon/8(8+n), 0, 0)$;
- (iii) Pure Ising fixed point:⁹ $(u^*, \Delta^*, v^*) = (0, 0, \epsilon/72)$;
- (iv) Pure cubic fixed point:⁹ $(u^*, \Delta^*, v^*) = (\epsilon/24n, 0, (n-4)\epsilon/72n)$;
- (v) Random Heisenberg fixed point: $(u^*, \Delta^*, v^*) = (\epsilon/32(n-1), (4-n)\epsilon/128(n-1), 0)$;
- (vi) Random cubic fixed point: $(u^*, \Delta^*, v^*) = (1/48(n-2), (4-n)/192(n-2), (n-4)/144(n-2))$;
- (vii) Nonphysical fixed point: $(u^*, \Delta^*, v^*) = (0, -\epsilon/64, 0)$.

The nonphysically of the last fixed point is due to the fact that the value of Δ^* is negative; on the other hand, being a mean square value of quenched disorder fluctuations, it is only positive defined¹¹.

Notice that the random Heisenberg fixed point is physically meaningful for $4 \geq n > 1$, and the random cubic fixed point is physically meaningful for $4 \geq n > 2$.

To analyze stability of the fixed points we assume

$$u = u^* + \delta u, \quad \Delta = \Delta^* + \delta \Delta, \quad v = v^* + \delta v \quad (53)$$

and linearize the RG equations.

In the vicinity of the Gaussian fixed point we obtain

$$\frac{d}{d\ell} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix} = \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix}. \quad (54)$$

The Gaussian fixed point is stable for $d > 4$.

In the vicinity of the pure Heisenberg fixed point we obtain

$$\frac{d}{d\ell} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix} = \frac{\epsilon}{8+n} \begin{pmatrix} -(8+n) & 12 & -6 \\ 0 & 4-n & 0 \\ 0 & 0 & n-4 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix}. \quad (55)$$

This fixed point is unstable both for $n > 4$ and $n < 4$.

In the vicinity of the pure Ising fixed point we obtain

$$\frac{d}{d\ell} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix} = \frac{\epsilon}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & -6 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix}. \quad (56)$$

This fixed point is always unstable.

In the vicinity of the pure cubic fixed point we obtain

$$\begin{aligned} \frac{d}{d\ell} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix} \\ = \frac{\epsilon}{3n} \begin{pmatrix} -(8+n) & 12 & -6 \\ 0 & 4-n & 0 \\ 4(4-n) & -4(4-n) & 3(4-n) \end{pmatrix} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix}. \end{aligned} \quad (57)$$

The three eigenvalues μ_1 , μ_2 and μ_3 of the system (57) are

$$\mu_1 = -\epsilon, \quad \mu_2 = \frac{(4-n)\epsilon}{3n}, \quad \mu_3 = 4-n. \quad (58)$$

This fixed point is stable for $d < 4$ and $n > 4$.

In the vicinity of the random Heisenberg fixed point we obtain

$$\begin{aligned} \frac{d}{d\ell} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix} &= \frac{\epsilon}{4(n-1)} \\ &\begin{pmatrix} -(8+n) & 12 & -6/(8+n) \\ (1+n/2)(n-4) & 2(4-n) & -6/(8+n) \\ 0 & 0 & n-4 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix}. \end{aligned} \quad (59)$$

The three eigenvalues of the system (59) μ_1 , μ_2 and μ_3 are

$$\mu_1 = -\epsilon, \quad \mu_2 = \frac{(n-4)\epsilon}{4(n-1)}, \quad \mu_3 = n-4. \quad (60)$$

Hence the fixed point is stable for $d < 4$ and $n < 4$.

In the vicinity of the random cubic fixed point we obtain

$$\begin{aligned} \frac{d}{d\ell} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix} &= \begin{pmatrix} -8(8+n)u^* & 96u^* & -48u^* \\ -16(2+n)\Delta^* & 64\Delta^* & -48\Delta^* \\ -96v^* & 96v^* & -72v^* \end{pmatrix} \\ &\cdot \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix} = \frac{\epsilon}{12(n-2)} \\ &\begin{pmatrix} -(8+n) & 24 & -12 \\ (2+n)(n-4) & 4(4-n) & 3(n-4) \\ 8(4-n) & 8(n-4) & 6(4-n) \end{pmatrix} \begin{pmatrix} \delta u \\ \delta \Delta \\ \delta v \end{pmatrix}. \end{aligned} \quad (61)$$

From the fact that determinant of the matrix in Eq. (61) is positive we come to the conclusion that at least one of the eigenvalues of the matrix is positive, hence the random cubic fixed point is unstable.

Phase portraits of the system (27),(28), (29) (for $p = 0$) for $d = 3$ are presented: without cubic anisotropy ($v = 0$) – on Fig. 1, and without disorder ($\Delta = 0$) – on Fig. 2. We see the stable random Heisenberg and pure cubic fixed points for $n < 4$ and $n > 4$ respectively, and the unstable Gaussian and pure Ising fixed points.

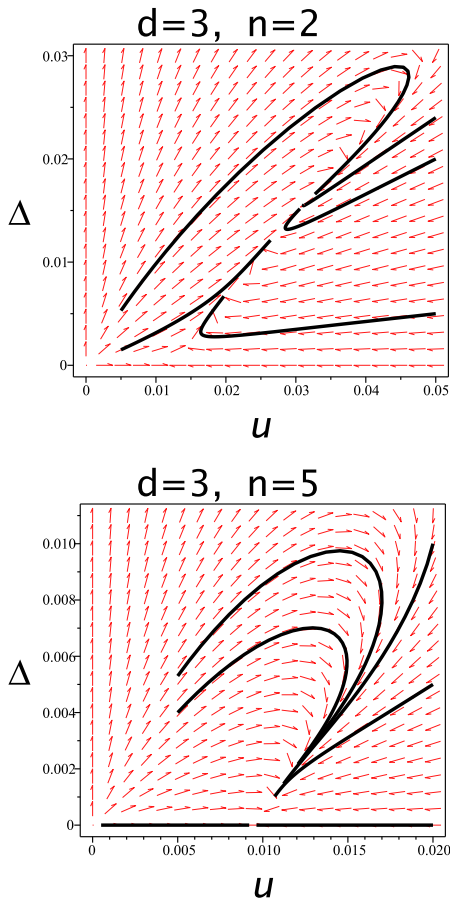


FIG. 1: Phase portraits of the RG equations for $O(n)$ symmetric ϕ^4 model in d dimensions with quenched disorder (Eqs. (27),(28) with $p = 0$ and $v = 0$).

The critical exponent ν is found from the RG equation for τ :¹⁰

$$\begin{aligned} \nu^{-1} &= \lambda_\tau = \left. \frac{d \ln \tau}{d \ell} \right|_{(u, \Delta, v) = (u^*, \Delta^*, v^*)} \\ &= 2 - 8 [(2+n)u^* + 3v^* - 2\Delta^*]. \end{aligned} \quad (62)$$

For the random Heisenberg fixed point $\nu_R = \frac{1}{2} + \frac{3n\epsilon}{32(n-1)}$, and for the pure cubic fixed point $\nu_{PC} = \frac{1}{2} + \frac{(n-1)\epsilon}{6n}$. Since

in linear to ϵ approximation there is no renormalization of the gradient term in Eq. (9), for all fixed points $\eta = 0$. Other critical exponents can be obtained from the two calculated ones using scaling relations⁵.

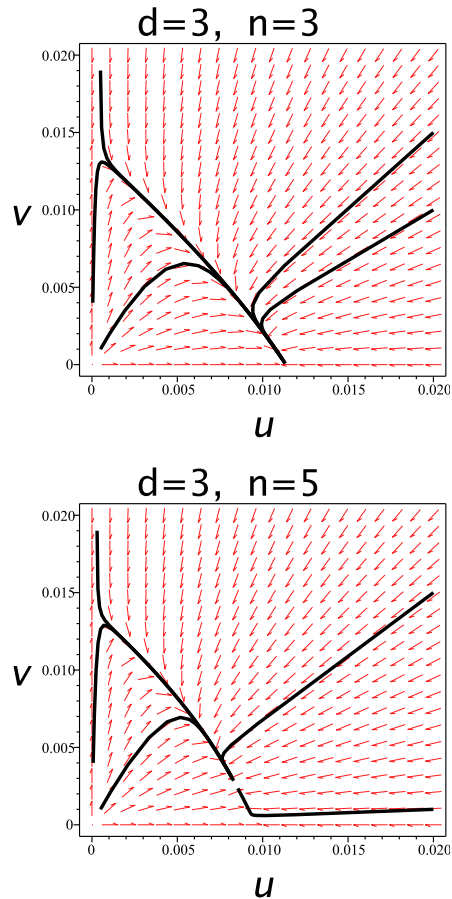


FIG. 2: Phase portraits of the RG equations for pure ϕ^4 model with cubic anisotropy in d dimensions (Eqs. (27),(29) with $p = 0$ and $\Delta = 0$).

VII. ACKNOWLEDGEMENTS

We see our modest contribution as one more illustration to the famous saying of Leopold Kronecker: ‘Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk’ (‘God made the integers, all else is the work of man’).

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